

Thursday 27th May 2004, 10 am

Time Allowed: 3 hours

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given and the beginning of each Section.

Calculators are not permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Section A: *You should attempt all questions. Marks awarded are shown next to the questions.*

1. Consider a sphere of radius 3 centred at the origin.
 - (a) Express the surface of the sphere in the form $f(x, y, z) = 0$.
 - (b) Find a plane tangent to the sphere at the point $\mathbf{r}_1 = (1, 2, 2)$.
 - (c) Find the position vector for a line through $\mathbf{r}_1$ and $\mathbf{r}_2 = (2, 1, 2)$.
 - (d) Find the position vector of a line through $\mathbf{r}_1$ and normal to the sphere. [10]
2. In the accompanying figure, the left and right panels illustrate two different vector fields.

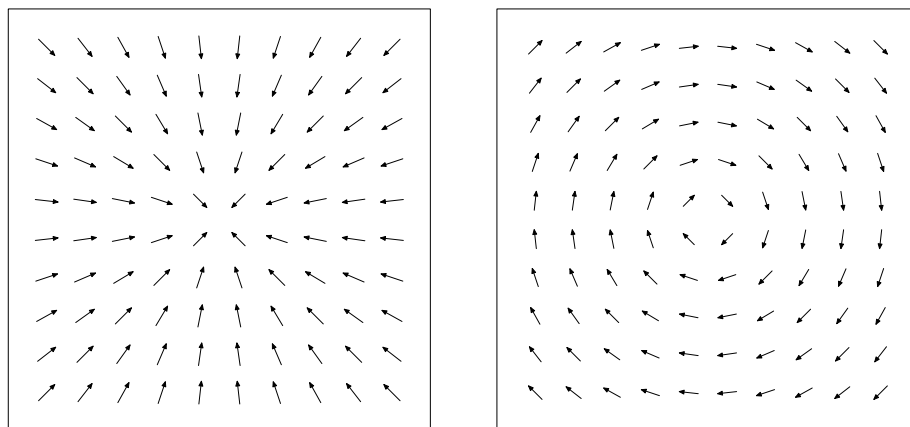


Figure for question 2.

For each panel, briefly explain whether the vector field has zero/nonzero divergence and zero/nonzero curl. [6]

3. Consider a line integral $\int \mathbf{F} \cdot d\mathbf{r}$ of each of the following vector fields.
 - (a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

- (b) $\mathbf{F} = x\mathbf{i} + z\mathbf{j} + y\mathbf{k}$
 (c) $\mathbf{F} = x\mathbf{i} + z\mathbf{j} - y\mathbf{k}$
 (d) $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

In each case, state whether the line integral will depend on the path. Give brief reasons in your answer. [8]

4. Simplify the following expressions.

- (a) $\oint \mathbf{r} \cdot d\mathbf{S}$ over an arbitrary closed surface of volume V .
 (b) $\oint \mathbf{r} \cdot d\mathbf{r}$ over an arbitrary closed curve.

You may use standard results with brief explanations. [6]

5. Write the expression

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D})$$

using index notation. Which indices in your expression are free and which are dummy? Simplify your expression to one that equals

$$(\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$$

You may assume standard index identities. [10]

6. Suppose the following functions are Fourier-expanded with $-\pi < x < \pi$.

- (a) x
 (b) $\sin^2 x$
 (c) $P_2(x/\pi)$ [Legendre polynomial].

In each case, state whether the Fourier series will have only $\sin nx$ terms, or only $\cos nx$ terms, or both $\sin nx$ and $\cos nx$ terms. Also state whether the constant term will be nonzero.

Give brief reasons in your answer. It is not necessary to calculate the Fourier coefficients. [7]

7. A scalar field satisfies Laplace's equation and has the form $\Phi(x, y) = f(y) \sin x$. Write down a general solution for $f(y)$.

Now consider the square $x, y \in [0, \pi]$. We impose the boundary conditions on the sides of the square

$$\Phi(x, y) = \begin{cases} \sin x & \text{for } y = \pi \\ 0 & \text{for } y = 0, x = 0, \text{ and } x = \pi \end{cases}$$

What is the particular $f(y)$ that will make $\Phi(x, y)$ satisfy these boundary conditions? [6]

8. We want to find a path $y(x)$ between two given endpoints such that

$$\int (y'^2 - y^2) dx$$

is extremized. [Here y' means dy/dx as usual.] Determine the Euler-Lagrange equation for the problem and its general solution.

What is the particular solution if the endpoints are $(0, 0)$ and $(\frac{\pi}{2}, \frac{\pi}{2})$? [7]

Section B: Each question carries 20 marks. You may attempt all questions but only marks for the best TWO questions will be counted.

9. Show that about 4% of the Earth's surface lies north of the Arctic circle.

[Recall that the Arctic circle has latitude of about 66° . You may use standard results on coordinate systems, but be sure to explain your coordinate system and calculations clearly. You may also use a suitable approximation for $\sin x$.]

10. The vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ in the accompanying figure are the spherical polar unit vectors $\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi$ but not necessarily in the same order.

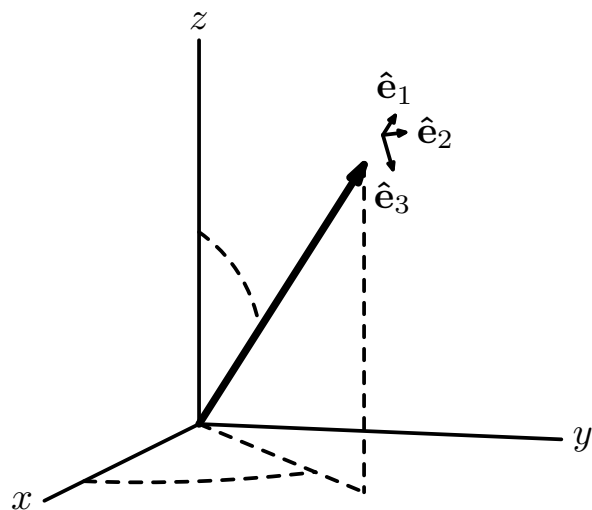


Figure for question 10.

- Match up $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ correctly with $\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi$, giving reasons briefly.
- Starting from the formulas defining (r, θ, ϕ) derive the relations

$$\begin{pmatrix} \hat{\mathbf{e}}_r \\ \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_\phi \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix}$$

and hence verify that $\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi$ are mutually perpendicular.

11. For a well-behaved vector field $\mathbf{F}$ show that $\nabla \cdot \nabla \times \mathbf{F} = 0$ *without* any coordinate-dependent calculation.

Hint: One possible solution is to consider two surfaces enclosed by the same closed curve and invoke the divergence and Stokes' theorems with suitable arguments. You may wish to use a sketch.

12. Expand

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

in a Fourier series for $-\pi < x < \pi$. Use the series to derive an infinite series for π .

13. A body moves from $(0, 0)$ to $(\frac{\pi}{2} - 1, 1)$ along the curve

$$x = (u - \sin u) \quad y = (1 - \cos u)$$

Assuming that the speed of the body at any (x, y) is $\sqrt{y}$, show that this particular curve extremizes the travel time.

Calculus III

Answers

1. {Unseen, but several similar examples seen.}

(a) $x^2 + y^2 + z^2 - 9 = 0.$ [2]

Taking the gradient $\mathbf{r}_1$ gives a vector normal to the surface

$$\mathbf{p} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

[2]

(b) The tangent plane is $(\mathbf{r} - \mathbf{r}_1) \cdot \mathbf{p} = 0$. Thus

$$[(x - 1)\mathbf{i} + (y - 2)\mathbf{j} + (z - 2)\mathbf{k}] \cdot [\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}] = 0$$

or

$$x + 2y + 2z = 9$$
 [2]

(c) $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + s(\mathbf{i} - \mathbf{j})$ where $-\infty < s < \infty$. [2]

(d) $\mathbf{r} = s(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ where $-\infty < s < \infty$. [2]

2. {Seen in lectures. Similar examples also seen.}

Left panel: Arrows indicate a flux into the centre, so nonzero divergence. No circulation noticeable, so zero curl. [3]

Right panel: No sources or sinks noticeable, so zero divergence. Loops around origin will clearly have a nonzero line integral, hence nonzero curl. [3]

3. {Specific examples not all seen, but this type of question seen in lectures and test.}

(a) Path-independent because $\mathbf{F} = \frac{1}{2}\nabla(x^2 + y^2 + z^2)$ and $\nabla \times \mathbf{F} = 0$. {Either reason acceptable} [2]

(b) Path-independent because $\mathbf{F} = \nabla(\frac{1}{2}x^2 + yz)$ and $\nabla \times \mathbf{F} = 0$. [2]

(c) Path-dependent because $\nabla \times \mathbf{F} = -2\mathbf{i}$. [2]

(d) Path-independent because $\mathbf{F} = \nabla xyz$ and $\nabla \times \mathbf{F} = 0$. [2]

4. {Part (a) seen in coursework, part (b) seen in lectures.}

(a) Using the divergence theorem

$$\oint \mathbf{r} \cdot d\mathbf{S} = \int (\nabla \cdot \mathbf{r}) dV = 3V$$
 [3]

(b) Using Stokes theorem

$$\oint \mathbf{r} \cdot d\mathbf{r} = \int (\nabla \times \mathbf{r}) \cdot d\mathbf{S} = 0$$
 [3]

We have used $\nabla \cdot \mathbf{r} = 3$ and $\nabla \times \mathbf{r} = 0$.

5. {Coursework problem.}

The first expression is

$$\epsilon_{ijk} A_j B_k \epsilon_{imn} C_m D_n \quad [2]$$

All indices are dummy. [2]

We apply the standard two- ϵ identity {essential formulas to be learnt were highlighted in boxes in notes and emphasized in lectures} to get

$$(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})A_j B_k C_m D_n \quad [2]$$

which then simplifies to

$$A_j B_k C_j D_k - A_j B_k C_k D_j \quad [3]$$

which we identify the second expression. [1]

6. {Similar question seen. Fourier expansions of x and $\sin^2 x$ also seen. $P_2(x)$ was a boxed formula.}

(a) Odd function. so only sine terms, no constant or cosine terms. [2]

(b) Even function with nonzero mean. So constant and cosine terms, no sine terms. [2]

(c) $P_2(u) = \frac{1}{2}(3u^2 - 1)$ is an even function with zero mean in $[-1, 1]$. So only cosine terms, no constant or sine terms. [3]

7. {Unseen, but similar questions seen in coursework and tests.}

Must be exponential/hyperbolic with 1 as the coefficient of y .

$$f(y) = A \sinh(y + B) \quad [3]$$

The $x = 0$ and $x = \pi$ boundary conditions are already satisfied by $\sin x$. The $y = 0$ condition forces $B = 0$, and the $y = \pi$ condition gives A . Hence

$$f(y) = \frac{\sinh y}{\sinh \pi} \quad [3]$$

8. {General solution seen, these boundary conditions unseen.}

The Euler-Lagrange equation is

$$\frac{dy'}{dx} + y = 0$$

or

$$y'' + y = 0 \quad [3]$$

The well-known general solution is

$$y = A \sin(x + B) \quad [2]$$

Boundary conditions: the origin fixes $B = 0$ and the other boundary condition gives A . Hence

$$y = \frac{\pi}{2} \sin x \quad [2]$$

9. {Seen up to the $\sin^2 12^\circ$ stage. The numerical part is unseen.}

We set up spherical polars with equator being on the x, y plane and the Earth's axis being the z axis. The Arctic circle is then at $\theta = 24^\circ$. [3]

The area from $\theta = 0$ to $\theta = \theta_1$ is

$$r^2 \int_0^{2\pi} d\phi \int_0^{\theta_1} \sin \theta d\theta$$

where r is the Earth's radius. [3]

Integrating gives

$$2\pi r^2(1 - \cos \theta_1) = 4\pi r^2 \sin^2 \theta_1 \quad [4]$$

For the whole area we take $\theta_1 = \pi$ giving $4\pi r^2$. Hence the fractional area is $\sin^2 \theta_1$. Thus the fraction north of the Arctic circle is $\sin^2 12^\circ$. [2]

Since 12° is a small angle, we apply $\sin^2 x \simeq x^2$. {My feeling is students unused to arithmetic without calculators will find this step non-trivial.} [4]

We have $12^\circ = \frac{1}{15}\pi$ radians $\simeq 0.2$, resulting in 4%. [4]

10. {Part (a) is seen in lectures and a test. Part (b) is bookwork— some tedious algebraic steps are omitted below.}

(a) Referring to the standard spherical polar coordinates, the figure shows $\hat{\mathbf{e}}_1$ in the direction of increasing r , $\hat{\mathbf{e}}_2$ in the direction of increasing ϕ , and $\hat{\mathbf{e}}_3$ in the direction of increasing θ . Hence $\hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_\phi$, $\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_\theta$. [6]

(b) From the definitions {boxed formula in notes}

$$x = r \sin \theta \cos \phi \quad y = r \sin \theta \sin \phi \quad z = r \cos \theta$$

we have

$$\mathbf{r} = r \sin \theta \cos \phi \mathbf{i} + r \sin \theta \sin \phi \mathbf{j} + r \cos \theta \mathbf{k} \quad [3]$$

By definition

$$\begin{aligned} \hat{\mathbf{e}}_r &= \frac{\partial \mathbf{r}}{\partial r} \left| \frac{\partial \mathbf{r}}{\partial r} \right|^{-1} \\ \hat{\mathbf{e}}_\theta &= \frac{\partial \mathbf{r}}{\partial \theta} \left| \frac{\partial \mathbf{r}}{\partial \theta} \right|^{-1} \\ \hat{\mathbf{e}}_\phi &= \frac{\partial \mathbf{r}}{\partial \phi} \left| \frac{\partial \mathbf{r}}{\partial \phi} \right|^{-1} \end{aligned} \quad [6]$$

and working out the derivatives and simplifying we get the result.

For orthogonality we take mutual dot products. [2]

We see

$$\hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_\theta \cdot \hat{\mathbf{e}}_\phi = \hat{\mathbf{e}}_\phi \cdot \hat{\mathbf{e}}_r = 0$$

through cancellation. [3]

11. {Coursework problem.}

Consider the northern and southern hemispheres enclosed by the equator. By Stokes' theorem, for any vector field $\mathbf{F}$

$$\oint_{\text{eq}} \mathbf{F} \cdot d\mathbf{r} = \int_N \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$$

If the equator integral goes anti-clockwise, the northern hemisphere will have its normal directed outwards while the southern hemisphere will have its normal directed inwards.

[The idea here is using Stokes' theorem to get two equal curl things.] [8]

If we take both normals outwards

$$\int_N \nabla \times \mathbf{F} \cdot d\mathbf{S} + \int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = 0.$$

But now the integral is over a closed surface, hence by the divergence theorem

$$\int (\nabla \cdot \nabla \times \mathbf{F}) dV = 0$$

[The idea here is using the divergence theorem to get zero for the divergence.] [8]

Since the whole argument applies for any sphere

$$\nabla \cdot \nabla \times \mathbf{F} = 0$$

identically. [4]

12. {Standard example, covered in lectures and notes. Expansion formulas were boxed in notes.}

For our Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

we have

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \quad [2]$$

which gives

$$a_0 = \frac{1}{2} \quad a_{n>0} = 0 \quad [2]$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \quad [2]$$

which gives

$$\begin{aligned} b_n &= \frac{1 - \cos n\pi}{n\pi} \quad n > 0 \\ &= \frac{2}{n\pi} \quad n \text{ odd} \end{aligned} \quad [6]$$

Collecting all nonzero terms we have

$$f(x) = \frac{1}{2} + 2 \sum_{n \text{ odd}} \frac{\sin nx}{n\pi} \quad [2]$$

Now we put $x = -\frac{\pi}{2}$. Note that $\sin x = -1, \sin 2x = 0, \sin 3x = 1, \sin 4x = 0$. [4]

This gives us

$$0 = \frac{1}{2} - \frac{2}{\pi} + \frac{2}{3\pi} - \frac{2}{5\pi} + \frac{2}{7\pi} + \dots$$

or

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \quad [2]$$

Other substitutions are also possible.

13. {Classic problem, slight variant seen in coursework.}

The time is arclength/velocity, hence

$$\int_0^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{y}} dx \quad [4]$$

Simplest to use the Hamiltonian form of the Euler-Lagrange equations, i.e., to extremize $\int L dx$ we need {boxed formula in notes}

$$y' \frac{\partial L}{\partial y'} - L = \text{const} \quad [2]$$

Applying this we get

$$\frac{1}{\sqrt{y}\sqrt{1+y'^2}} = \text{const}$$

which we can rewrite as

$$y(1+y'^2) = \text{another const} \quad [6]$$

By trigonometric identities

$$y = 2 \sin^2 \frac{1}{2}u, \quad y' = \cot \frac{1}{2}u \Rightarrow y(1+y'^2) = 2 \quad [6]$$

so the given solution satisfies the differential equation.

Going from $u = 0$ to $\frac{\pi}{2}$ the given solution also satisfies the boundary conditions. [2]